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On the Reduction of Pseudo-Differential Operators to Canonical Forms

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INTRODUCTION

In this paper, we are concerned with microlocal reductions of classical pseudo-differential operators with real principal symbols near a double characteristic point. We obtain various simple forms of the lower order terms of the operators, according to the nature (symplectic or involutive) of the manifold of double characteristic points.

Results of this type have also been obtained by Ivrii [15] and Hanges [10] (in the hyperbolic non-involutive case), Uhlmann [21] (in the hyperbolic involutive case), Guillemin and Shaeffer [9] (for some Fuchsian-type operators), Duistermaat-Sjöstrand [6] (for complex symbols) and Kashiwara-Kawai-Oshima [16] (in the complex domain).

I. GENERALITIES

1. NOTATIONS; STATEMENT OF THE PROBLEM

Let Ω be an open set in $\mathbb{R}^N$, and n a point of $T^*(\Omega)$.

As usual, a pseudo-differential operator $P(x, D_x)$ is called "classical" of order m if its full symbol $p(x, \xi)$ has an asymptotic expansion of the form $p \sim p_0 + p_1 + \cdots + p_k + \cdots$ (when $|\xi| \to +\infty$), with p_k homogeneous of order $m - k$ and $k \in \mathbb{N}$, $m \in \mathbb{R}$.

We will say that P and Q are "the same" near n if $p - q$ is rapidly decreasing in a conic neighborhood of n.

In the whole paper, we are dealing with classical pseudo differential operators near n; in particular, we will always assume, if necessary, that our operators are properly supported.

Let P and Q be operators given near $n \in T^*(\Omega)$ and $n' \in T^*(\Omega')$ respectively: we are looking for elliptic Fourier-integral operators F and G such that $QF = GP$ near n; precisely let χ be a symplectic diffeomorphism, homogeneous of degree 1,

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from a conical neighborhood of n into a conical neighborhood of n', with $\chi(n)=n'$; we say that F: $C_0^\infty(\Omega) \to \mathscr{D}'(\Omega')$ is elliptic with canonical mapping χ near n if $\Lambda' =$ graph of χ near (n, n') (where Λ denotes the Lagrangean manifold of the distribution kernel of F) and the principal symbol of F is non-zero at (n, n'). In the equality $QF = GP$, F and G are assumed to have the same mapping χ.

If this equality holds then $q_0 \circ \chi = ep_0$, where e is an elliptic symbol near n. For given p_0, q_0, the existence of such χ and e has been studied by Hörmander [11], Sato [15], Sjöstrand [17], Sjöstrand-Duistermaat [6], Boutet de Monvel [5], Guillemin-Shaeffer [9] and many others.

Here, we assume this existence, and emphasize the role played by lower order terms; thus, the problem is reduced to the case $\Omega = \Omega'$, $n = n'$, $p_0 = q_0$.

This problem is clearly void for p_0 elliptic at n, and well-known if $p_0 \in \mathbb{R}$ and the Hamiltonian field H_{p_0} of p_0 at n is non-parallel to the cone axis. In the whole work, we restrict ourselves to the case: p_0 real, n is at most a double zero of p_0.

2. Two types of equivalence

If P and Q are defined near n, they will be called equivalent if there exist elliptic (classical) pseudo-differential operators A and B near n such that $AP = QB$; we will write $P \sim Q$.

Let $\mathscr{F}(p_0)$ be the set of all symplectic mappings χ (as above) such that $\chi(n) = n$, $p_0 \circ \chi = ep_0$, where e is elliptic and homogenous of order 0 near n. We call P and Q with $p_0 = q_0$, "Fourier-equivalent" (denoted by $P \overset{\mathscr{F}}{\sim} Q$) if there exist $\chi \in \mathscr{F}(p_0)$ and an elliptic Fourier integral operator F with canonical mapping χ such that $F^{-1}PF \sim Q$ (F^{-1} denotes any local inverse of F near n).

The following simple lemma shows the role played by the $\mathscr{F}$-equivalence.

LEMMA. *Let P near n, Q near n' be as in the non-reduced problem, and let χ_i (with $\chi_i(n) = n'$) and e_i be such that $q_0 \circ \chi_i = e_i p_0$ $(i = 1, 2)$. Denote by F_i an elliptic operator with canonical mapping χ_i, by E_i an elliptic pseudo-differential operator with principal symbol $1/e_i$ $(i = 1, 2)$. Then $E_1F_1^{-1}QF_1 \overset{\mathscr{F}(p_0)}{\sim} E_2F_2^{-1}QF_2$.*

II. RESULTS AND EXAMPLES

Let Σ be the set of double characteristics of P, and assume that Σ is a (conic) manifold, locally defined near n by the equations $u_1 = \cdots = u_l = 0$; then, for any function f near Σ, we denote by $\sum_\alpha f^{\alpha_1,\ldots,\alpha_l} u_1^{\alpha_1} \cdots u_l^{\alpha_l}$ the formal Taylor expansion of f with respect to $u_1, \ldots, u_l$. The other notations are explained in *I*.

In the following theorems, P and Q are as explained in Part I, with $p_0 = q_0$.

1. Involutive case (codimension $\Sigma = 1$)

Assume Σ of codimension 1: then it is no restriction (see [11]) to take $p_0 = \tau^2$ (the coordinates being (x, t, ξ, τ)) and $n = (0, 0, \xi_0 \neq 0, 0)$. With this choice of p_0, we have the following theorems.

THEOREM 1.1. [A] (i) *There exist* $\tilde{P}, \tilde{p}_0 = p_0$, $P \sim \tilde{P}$, *and* $\tilde{p}_j = \tilde{p}_j^0$ *for* $j \geqslant 1$.

(ii) $P \sim Q \to p_1^0 = q_1^0$.

(iii) *If* $p_1^0 = q_1^0 \neq 0$, *then* $P \sim Q$.

[B] *Assume that there exist a symplectic diffeomorphism* $\psi: (x, \xi) \mapsto (X(x, \xi), \Xi(x, \xi))$, *homogeneous of degree* 1, *with* $\psi(0, \xi_0) = (0, \xi_0)$, *and a smooth function* $\lambda(x, \xi)$, *homogeneous of degree zero*, $\lambda \in \mathbb{C}$, $\lambda \neq 0$, *such that, near* n, $\lambda(x, \xi) p_1^0(X(x,\xi), \Xi(x, \xi), t) > 0$, $\lambda(x, \xi)\, q_1^0(x, \xi, t) > 0$. *Then* $P \overset{\mathscr{F}}{\sim} Q$.

Example 1. An operator $P = D_t^2 + p_1 + p_2 + \cdots$ of Schrödinger type, i.e. with $p_1^0 < 0$ near n, can be reduced to $D_t^2 - D_{x_1}$ near $\xi = (1, 0, \ldots, 0)$, $\tau = 0$. This may allow explicit computations (compare with [3], [16], for instance).

Example 2. Consider $P = D_t^2 + p_1 + \cdots, Q = D_t^2 + q_1 + \cdots$ and assume that both p_1^0 and q_1^0 are real and non-zero. Then $P \overset{\mathscr{F}}{\sim} Q$ if and only if they are both hypoelliptic or both non-hypoelliptic near n. This is a consequence of the fact that, in our case P is hypoelliptic near n if and only if $p_1^0 > 0$ (see, for instance, [14]).

THEOREM 1.2. *Assume that* $p_1^0 \equiv 0$ *(Levi Condition). Then*:

[A] (i) $P \sim Q \Rightarrow p_1^0 = q_1^0 \equiv 0$.

(ii) *Set* $I(P) = p_2^0 - \frac{1}{4}(p_1^1)^2 + (i/2)\, p_{1t}^1$. *Then* $P \sim Q \Leftrightarrow I(P) = I(Q)$.

(iii) $I(P) \equiv 0 \Leftrightarrow P \sim H^2$, *where* H *is any first order operator with principal symbol* τ.

[B] *There exist* $\lambda(x, t, \xi)$, *real, homogeneous of order* 0, *such that* $P \overset{\mathscr{F}}{\sim} D_t^2 + i\lambda(x, t, D_x)$. *If* $I(P)$ *is real, then* $P \overset{\mathscr{F}}{\sim} D_t^2$.

Example 3. Let $P = \partial_t^2 + a\partial_t + b$, a and b real and of order 0. Then $P \overset{\mathscr{F}}{\sim} \partial_t^2$. On the other hand, $\partial_t^2 + i\lambda$ and $\partial_t^2 + i\mu$ $(\lambda, \mu \in \mathbb{R})$ are $\mathscr{F}$-equivalent if and only if λ and μ have the same sign.

2. Symplectic case (codim $\Sigma = 2$)

Assume Σ symplectic of codimension 2; it is no restriction to take $u_1 = t$, $u_2 = \tau$ in a space with coordinates (x, t, ξ, τ). In the hyperbolic case, we will take $p_0 = t\tau$, otherwise $p_0 = \tau^2 + t^2\xi_1^2$ (with $\xi_1 \neq 0$). We will use the following definition.

DEFINITION. Let f be a complex symbol homogeneous of degree d. We will say that H_f is H-solvable if, for any smooth v homogeneous of degree l, there exist a smooth u homogeneous of degree $l - d + 1$ such that $H_f u = v$. (See [12] for examples).

THEOREM 2.1. *Assume* $p_0 = t\tau = q_0$.

[A] (i) *There exists* $\tilde{P}$, $\tilde{p}_0 = p_0$, $P \sim \tilde{P}$, $\tilde{p}_j \sim \tilde{p}_j^0$ *for* $j \geqslant 1$ (*see* [10]).

(ii) $P \sim Q \Rightarrow p_1^0 = q_1^0$. *This is sufficient if* $H_{p_1^0}$ *is H-solvable.*

(iii) *If* $P \sim Q$ *and* $p_1^0 \equiv cte$, *then* $I(P) = I(Q)$, *where* $I(P) = p_2^0 - p_1^0 p_1^{1,1} - p_1^{1,0} p_1^{0,1}$.

(iv) *Assume* $p = p_0 + p_1 + p_k + p_{k+1} + \cdots$, $q = p_0 + p_1 + q_k + q_{k+1} + \cdots$, $p_1 = cte$, $k \geqslant 2$, *and both* P *and* Q *are tangential* (*i.e.* $p_l = p_l^0$, $q_l = q_l^0$, $l \geqslant 1$). *Then* $P \sim Q \Rightarrow p_k = q_k$, *and this is sufficient if* H_{p_k} *is H-solvable.*

[B] *Assume* $p \overset{\mathscr{F}}{\sim} Q$: *then there exist a symplectic mapping* $\chi(n) = n$, *such that* $q_1^0 = p_1^0 \circ \chi$ *or* $q_1^0 = -i - p_i^0 \circ \chi$. *Either one of these two conditions is sufficient if* $H_{p_1^0}$ *is H-solvable.*

Example 4. The operators tD_t and $tD_t + q_k$ are non-equivalent for any $k \geqslant 2$, $q_k \neq 0$, which proves that equivalence may involve terms of any order in the symbols of P and Q.

Example 5. Consider $P = D_t^2 - t^2 \mid D_x \mid^2 + p_1 + p_2 + \cdots$ near a point $n(x = 0, t = 0, \xi \neq 0, \tau = 0)$, which can be reduced to the form in Theorem 2.1. Then $\operatorname{Re} p_1^0$ determines the microlocal regularity of solutions in the kernel of P, while the jets of p_1, p_2 etc... on Σ determine how the singularities of a solution "branch" on Σ (see [1], [2]).

THEOREM 2.2. *Assume* $p_0 = q_0 = \tau^2 + t^2\xi_1^2$ $(\xi_1 \neq 0)$. *The results are analogous to those in Theorem* 2.1. *In particular*:

[A] (i) *There exist* $\tilde{P}$, $\tilde{p}_0 = p_0$, $P \sim \tilde{P}$, *and* $\tilde{p}_j = \tilde{p}_j^0$ *for* $j \geqslant 1$.

(ii) $P \sim Q \Rightarrow p_1^0 = q_1^0$. *This is sufficient if* $H_{p_1^0 \backslash \xi_1}$ *is H-solvable.*

(iii) *If* P *and* Q *are both tangential,* $P \sim Q$ *and* $p_1^0/\xi_1 = cte$, *then* $p_2^0 = q_2^0$.

[B] *Assume* $P \overset{\mathscr{F}}{\sim} Q$: *then there exist a symplectic mapping* $\chi: \Sigma \to \Sigma$, $\chi(n) = n$, *such that* $(q_1^0/\xi_1) = (p_1^0/\xi_1) \circ \chi$. *This is sufficient if* $H_{p_1^0/\xi_1}$ *is H-solvable.*

Example 6. Consider $P = D_t^2 + t^2 D_1^2 + \lambda D_1 + \cdots$ $(\lambda \in \mathbb{C})$. It is well known that P is hypoelliptic unless λ belongs to a discrete set S of numbers (see, for instance [8]). If $\lambda \in S$, then the hypoellipticity of P may depend on a lower order term of P of any order (in analogy with example 5) (see [18]). On the other hand, λ cannot be changed by $\mathscr{F}$-equivalence, and for any value of λ, zero order terms are relevant for equivalence (by (iii)). This shows that equivalence by elliptic operators is too strict as far as hypoellipicity is concerned. In fact, the set of values S appears when using concatenations (see [18], [7], [4]).

3. Hyperbolic involutive case (codim $\Sigma = 2$)

Without restriction, we assume that $\Sigma = \{\xi = \eta = 0\}$, in a space with coordinates $(x, y, z, \xi, \eta, \zeta)$. We take $p_0 = \xi\eta$. In this case, many "invariants" of P and Q appear, and we have no sufficient condition for equivalence. An outline of further results is given in part III.

THEOREM 3.1. (i) *There exist* $\tilde{P}$, $\tilde{p}_0 = p_0$, $P \sim \tilde{P}$ *and* $\tilde{p}_j(x, y, z, \xi, \eta, \zeta) = \tilde{p}_j^0(x, y, z, \zeta) + \tilde{p}_j^1(x, y, z, \zeta)\xi, j \geqslant 1$.

(ii) *If* $P \sim Q$, *then* $p_1^0 = q_1^0$, $I_1(P) = I_1(Q)$, *where* $I_1(P) = (p_1^1)_x - (p_1^2)_y$.

(iii) *If* $P \sim Q$ *and* $p_1^0 = q_1^0 \equiv 0$ (*Levi Condition*), *then* (ii) *holds, and* $I_2(P) = I_2(Q)$, *with* $I_2(P) = p_2^1 - p_1^1 p_1^2 + i p_{1x}^1$.

III. PROOFS OF THE THEOREMS

1. General equations

We are looking for classical elliptic pseudo-differential operators A and B such that $AP = QB$; we denote the various symbols by small letters, with the expansions $a = a_0 + a_1 + \cdots$, $b = b_0 + b_1 + \cdots$, $p = p_0 + p_1 + \cdots$, $q = p_0 + q_1 + \cdots$.

We have $(A_0 + A_1 + \cdots)(P_0 + P_1 + \cdots) = (P_0 + Q_1 + \cdots)(B_0 + B_1 + \cdots)$, i.e.

$$\underbrace{A_0P_0 - P_0B_0}_{\mathrm{I}} + \underbrace{A_0P_1 - Q_1B_0 + A_1P_0 - P_0B_1}_{\mathrm{II}}$$
$$+ \underbrace{A_0P_2 - P_0B_2 + A_1P_1 - Q_1B_1 + A_2P_0 - P_0B_2}_{\mathrm{III}} + \cdots$$
$$= 0$$

where I is of order equal to order m of P and Q, II of order $m - 1$, III of order

$m-2$, and so on. This gives us, for the symbols, the successive equations

(0) $a_0 p_0 = p_0 b_0$, i.e. $a_0 = b_0$.

(1) $-\frac{1}{i} H_{p_0} a_0 + a_0(p_1 - q_1) + p_0(a_1 - b_1) = 0.$

(2) $$-\frac{1}{i}(p_{0\xi}\cdot b_{1x} - p_{0x}a_{1\xi}) + a_0(p_2 - q_2) + p_0(a_2 - b_2) + a_1 p_1 - q_1 b_1$$
$$+ \frac{1}{i}(a_{0\xi}\cdot p_{1x} - q_{1\xi}a_{0x})$$
$$-\frac{1}{2}\left(\sum_{|\alpha|=2} \frac{\partial^2 a_0}{\partial \xi^\alpha}\frac{\partial^2 p_0}{\partial x^\alpha} - \sum_{|\alpha|=2} \frac{\partial^2 a_0}{\partial x^\alpha}\frac{\partial^2 p_0}{\partial \xi^\alpha}\right) = 0,$$

and so on (here, the coordinates in the cotangent space are denoted by (x, ξ), and subscripts mean derivatives).

2. Operators with characteristics of constant multiplicity

We assume $p_0 = \tau^2$, in a space where the coordinates are (x, t, ξ, τ), near a point n of $\Sigma = \{\tau = 0\}$;

(a) The equation (1) is $-(2/i)\,\tau\partial_t a_0 + \tau^2(a_1 - b_1) + a_0(p_1 - q_1) = 0$, so that if $a_0 \neq 0$ exist, then $p_1{}^0 = q_1{}^0$. Let us assume now $p_1{}^0 = q_1{}^0$; giving $a_1 - b_1$ an arbitrary value, we can solve (1) with $a_0 \neq 0$. Considering the value of the left hand side of (2) on $\tau = 0$, we see that it can be made equal to zero, provided that $(p_2 - q_2)^0$ is conveniently chosen (because $a_0 \neq 0$); for this choice, giving $a_2 - b_2$ an arbitrary value, we can find a_1 solution of (2). Consider now (l), assuming that all $a_k, b_k (k \leqslant l-2)$, $a_k - b_k (k \leqslant l-1)$ and $q_k(k \leqslant l-1)$ are already chosen; then we can fix $(p_l - q_l)^0$ such that (l) can be solved in smooth functions, $a_l - b_l$ being arbitrary. This proves (i) and (ii), Theorem 1.1.

(b) Let us denote by $L(p)$ the left hand side of (p):

$$(L(2))^0 = a_0{}^0(p_2 - q_2)^0 + p_1{}^0(a_1 - b_1)^0$$
$$+ \frac{1}{i}(a^0_{0\xi}p^0_{1x} + a_0{}^1 p^0_{1t} - q^0_{1\xi}a^0_{0x} - q_1{}^1 a^0_{0t}) + a^0_{tt}\,.$$

From (1), $2i\partial_t a_0{}^0 + a_0{}^0(p_1 - q_1)^1 = 0,$

$$2i\partial_t a_0{}^1 + (a_1 - b_1)^0 + a_0{}^0(p_1 - q_1)^2 + a_0{}^1(p_1 - q_1)^1 = 0,$$

so that, equivalently,

$$(L(2))^0 = a_0{}^0[(p_2 - q_2)^0 - \tfrac{1}{2}q_1{}^1(p_1 - q_1)^1 - p_1{}^0(p_1 - q_1)^2]$$
$$+ a^0_{tt} - 2ip_1{}^0\partial_t a_0{}^1 - a_0{}^1[ip^0_{1t} - p_1{}^0(p_1 - q_1)^1] + \frac{1}{i}(a^0_{0\xi}p^0_{1x} - q^0_{1\xi}a^0_{0x}).$$

Assume that $p_1{}^0 \neq 0$ at n: for any choice of $a_0{}^0$, satisfying $L(1))^1 = 0$, we can find $a_0{}^1$ such that $(L(2))^0 = 0$; for this choice, put $a_0 = a_0{}^0 + \tau a_0{}^1$, and take $(a_1 - b_1)$ determined by $(L(1)) = 0$. Now, a_0 and $a_1 - b_1$ are fixed for the next computations.

Consider $(L(3))^0$; it contains the same terms as $(L(2))^0$, with $a_0{}^0$ replaced by $a_1{}^0$ and $a_0{}^1$ replaced by $a_1{}^1$, and other terms, which depend only on a_0, $a_1 - b_1$ and p, and are therefore already known. We give $a_1{}^0$ any value for which $(L(2))^1 = 0$; in fact, $(L(2))^1 = 2ib^0_{1t} + b_1{}^0(p_1 - q_1) +$ known quantity, so that it is possible. Finally, we can find $a_1{}^1$ such that $(L(3))^0 = 0$, by solving an ordinary equation as above. With these choices of $a_1{}^0$ and $a_1{}^1$, we put $a_1 = a_1{}^0 + \tau a_1{}^1$, and determine $a_2 - b_2$ from the equation (2). From now on, a_0, a_1, $a_1 - b_1$ and $a_2 - b_2$ are fixed, and we choose $a_2{}^0$ and $a_2{}^1$ such that $(L(4))^0 = 0$, $(L(3))^1 = 0$; we solve then (3) by taking $a_3 - b_3$ conveniently, and so on. This proves (iii) in Theorem 1.1.

Assume now that the Levi-condition is satisfied: $p_1{}^0 = q_1{}^0 \equiv 0$ on Σ. We have

$$(L(2))^0 = a_0{}^0[(p_2 - q_2)^0 - \tfrac{1}{2}q_1{}^1(p_1 - q_1)^1] + a^0_{0tt}\,.$$

But $2i\partial_t a_0{}^0 + a_0{}^0(p_1 - q_1)^1 = 0$ so that

$$a^0_{0tt} = \frac{-1}{2i}[(p_1 - q_1)^1 a^0_{0t} + a_0{}^0(p_1 - q_1)^1_t],$$

$$a^0_{tt} = \frac{i}{2}a_0{}^0(p_1 - q_1)^1_t - \frac{1}{4}a_0{}^0((p_1 - q_1)')^2, \ (L(2))^0$$
$$= a_0{}^0\left[(p_2 - q_2)^0 - \frac{1}{4}(p_1{}^1)^2 + \frac{1}{4}(q_1{}^1)^2 + \frac{i}{2}(p_1 - q_1)^1_t\right].$$

We set $I(P) = p_2{}^0 - \frac{1}{4}(p_1{}^1)^2 + (i/2)p^1_{1t}$, and obtain that if $P \sim Q$, then $(L(2))^0 = 0$, which implies $I(P) = I(Q)$, proving the first implication of (ii), Theorem 1.2.

In order to prove the converse implication, let us consider first

$$(L(3))^0 : (L(3))^0 = a_0{}^0(p_3 - q_3)^0 + a_1{}^0p_2{}^0 - q_2{}^0b_1{}^0 + \frac{1}{i}(a^0_{0\xi}p^0_{2x} - a^0_{0x}q^0_{2\xi})$$
$$+ \frac{1}{i}a_0{}^1p^0_{2t} - \frac{1}{i}a^0_{0t}q_2{}^1 - \frac{1}{i}q_1{}^1b^0_{1t} + b^0_{1tt} + q^1_{1\xi}a^0_{0xt} + q_1{}^2a^0_{tt}\,.$$

We have moreover

$$(L(2))^1 = a_0{}^0(p_2 - q_2)^1 + a_0{}^1(p_2 - q_2)^0 + a_1{}^0 p_1{}^1 - q_1{}^1 b_1{}^0 + 2ib_{1t}^0$$
$$+ \frac{1}{i}(a_{0\xi}^0 p_{1x}^1 - q_{1\xi}^1 a_{0x}^0) - \frac{1}{i} q_1{}^1 a_{0t}^1 + \frac{1}{i} a_0{}^1 p_1{}^1 t - \frac{1}{i} q_1{}^2 a_{0t}^0 + a_{0tt}^1,$$

and

$$(L(1))^2 = 2ia_{0t}^1 + (a_1 - b_1)^0 + a_0{}^0(p_1 - q_1)^2 + a_0{}^1(p_1 - q_1)^1,$$

so that

$$(L(2))^1 = a_0{}^1[I(P) - I(Q)] + a_0{}^0(p_2 - q_2)^1 + a_1{}^0 p_1{}^1 - q_1{}^1 b_1{}^0 + 2ib_{1t}^0$$
$$+ \frac{1}{i}(a_{0\xi}^0 p_{1x}^1 - q_{1\xi}^1 a_{0x}^0) - \frac{1}{2} q_1{}^1[(a_1 - b_1)^0 + a_0{}^0(p_1 - q_1)^2] - \frac{1}{i} q_1{}^2 a_{0t}^0$$
$$+ \frac{i}{2}[(a_1 - b_1)_t^0 + (a_0{}^0(p_1 - q_1)^2)_t]$$
$$- \frac{1}{4} q_1{}^1[(a_1 - b_1)^0 + a_0{}^0(p_1 - q_1)^2].$$

We have already the necessary condition $I(P) = I(Q)$, so that $(L(2))^1$ happens to be independent of $a_0{}^1$.

Now we assume that $a_0{}^0$ has already been chosen as a non-zero solution of $(L(1))^1 = 0$, and consider the system

$$(L(3))^0 = 0, \qquad (L(2))^1 = 0, \qquad (L(1))^2 = 0$$

as a system of ordinary differential equations (in t) in the unknown functions $a_0{}^1$, $b_1{}^0$ and $a_1{}^0$. This system can be reduced to a (4×4)-first-order linear system, which can be solved by standard process.

Having chosen a solution $a_0{}^1$, $b_1{}^0$ and $a_1{}^0$ of this system, we take $a_1 - b_1 = (a_1 - b_1)^0$, and solve (1) in the following way; let $a_0 = a_0{}^0 + \tau a_0{}^1 + \tau^2\alpha$, $a_0{}^0$ and $a_0{}^1$ having the values chosen above; then (1) reduces to $2i\tau(\partial_t a_0{}^0 + \tau\partial_t a_0{}^1 + \tau^2\partial_t\alpha) + \tau^2(a_1 - b_1)^0 + \tau a_0{}^0(p_1 - q_1)^1 + \tau^2(a_0{}^1(p_1 - q_1)^2 + a_0{}^1(p_1 - q_1)^1) + \tau^3(\alpha(p_1 - q_1)^1 + \text{known } q.) = 0$, i.e. $2i\partial_t\alpha + \alpha(p_1 - q_1)^1 + \text{known } q. = 0$; we take for α any solution of the last equation.

We have now the following situation: a_0, $a_1{}^0$, $b_1{}^0$ and $a_1 - b_1$ are chosen, (1) is solved. $(L(3))^0 = (L(2))^1 = (L(2))^0 = 0$.

To proceed further, we consider $(L(2))^2$, $(L(3))^1$ and $(L(4))^0$: $(L(2))^2 = 2ib_{1t}^1 + (a_2 - b_2)^0 + a_1{}^1 p_1 - q_1{}^1 b_1{}^1 + \text{known terms}$, $(L(3))^1$ is similar to $(L(2))^1$ (with

$a_1{}^0$ or $b_1{}^0$ instead of $a_0{}^0$, $b_1{}^1$ instead of $a_0{}^1$, $a_2{}^0$ and $b_2{}^0$ instead of $a_1{}^0$ and $b_1{}^0$) except for known terms (depending on a_0), and so is $(L(4))^0$:

$$\begin{aligned}(L(4))^0 = {} & a_1{}^0 p_3{}^0 - q_3{}^0 b_1{}^0 + a_2{}^0 p_2{}^0 - q_2{}^0 b_2{}^0 + \frac{1}{i}\,(a^0_{1\xi} p^0_{2x} - b^0_{1x} q^0_{2\xi}) \\ & + \frac{1}{i}\, a_1{}^1 p^0_{2t} - \frac{1}{i}\, b^0_{1t} q_2{}^1 \\ & - \frac{1}{i}\, q_1{}^1 b^0_{2t} + q^1_{1\xi} b^0_{1xt} + b^0_{2tt} + q^2 b^0_{1tt} + \text{terms depending on } a_0 .\end{aligned}$$

We have now a new system $(L(4))^0 = 0$, $(L(3))^1 = 0$, $(L(2))^2 = 0$ in the unknown functions $b_1{}^1$, $a_2{}^0$ and $b_2{}^0$. We solve it, and then solve (2) as an equation in $b_1{}^1$ with the previously determined values of $b_1{}^0$ and $b_1{}^1$, and $a_2 - b_2 = a_2{}^0 - b_2{}^0$.

Then a_0, a_1, b_1, $a_2 - b_2$, $a_2{}^0$, $b_2{}^0$ are known, (1) and (2) are solved, $(L(3))^0 = (L(3))^1 = (L(4))^0 = 0$, and we continue as above.

This completes the proof of (ii), Theorem 1.2.

(c) Set $H = D_t + \Lambda_0 + \Lambda_1 + \cdots$, where $\sigma(\Lambda_k) = \lambda_k$ is a symbol homogeneous of order $-k$. Then $H^2 = D_t{}^2 + D_t\Lambda_0 + \Lambda_0 D_t + D_t\Lambda_1 + \Lambda_0{}^2 + \Lambda_1 D_t + \cdots$, so that, for $P = H^2$, $p_0 = \tau^2$, $p_1 = 2\tau\lambda_0$, $p_2 = \lambda_0{}^2 + 2\tau\lambda_1 + (D_t\lambda_0)$, and $I(P) = (D_t\lambda_0{}^0) + (\lambda_0{}^0)^2 - \frac{1}{4}(2\lambda_0{}^0)^2 + \frac{1}{2}2\lambda_{0t} = 0$.

Conversely, take P satisfying the Levi Condition: $P = D_t{}^2 + CD_t + P_2 + \cdots$ where C is of order 0, and set $H = D_t + C/2$; then $P = H^2 + D_0 + D_1 + \cdots$, (where $\sigma(D_k) = d_k$ is a symbol homogeneous of order $-k$), and $I(P) = d_0{}^0$ (because $I(H^2) = 0$). If we assume $I(P) = 0$, then $P \sim H^2$, by (ii) of Theorem 1.2, which proves (ii).

(d) Let us consider now the equivalence by Fourier Integral operators.

First, we need a simple lemma.

Lemma. *Let $\Sigma = \{\tau = 0\}$. A smooth mapping $\chi\colon \Sigma \to \Sigma$, defined by $\chi(x, t, \xi) = (X(x, t, \xi), T(x, t, \xi), \Xi(x, t, \xi))$, can be extended to a symplectic mapping $\tilde{X}$ near Σ if and only if*

(a) *X and Ξ do not depend on t; $T_t{}^1 \neq 0$.*

(b) *The mapping $(x, \xi) \to ((X(x, \xi), \Xi(x, \xi))$ is symplectic.*

The standard proof is left to the reader.

Let us consider now an operator P, defined in a neighborhood of n, a symplectic mapping χ, homogeneous of degree 1, with $\chi(n) = n$, and $\chi\,|_\Sigma : \Sigma \to \Sigma$. Let F be a Fourier-integral operator with canonical transformation χ, elliptic near $(n, \chi(n)) = (n, n)$, and set $P^F = F^{-1}PF$. The principal part $p_0{}^F$ of P^F is then $p_0 \circ \chi$, that is $\mathscr{T}^2 = \tau^2\theta^2$. The subprincipal symbols are just composed by χ

at a point of Σ (see Hörmander [13]), that is, with the usual notation $(P^F)^S = P^S \circ \chi$. Here (with $p_0 = \tau^2$), this simply reads $(P_1{}^F)^0 = p_1{}^0 \circ (\chi\,|_\Sigma)$.

Let Θ be the elliptic operator of order 0 with symbol θ^{-2}, and $Q = \Theta P^F$; then Q and P have the same principal part $q_0 = p_0 = \tau^2$, and $q_1{}^0 = (1/\theta^2)\, p_1{}^0 \circ (\chi\,|_\Sigma) = T_t{}^2 p_1{}^0(\chi\,|_\Sigma)$ (notations of the lemma).

From the above formula for $q_1{}^0$, we get general but rather complicated necessary and sufficient conditions on $q_1{}^0$ and $p_1{}^0$ for Q and P to be $\mathscr{T}$-equivalent.

Let us consider the case B of Theorem 1.1: we have to solve the equation $\lambda(x, \xi)\, q_1{}^0(x, t, \xi) = T_t{}^2(x, t, \xi)\, p_1{}^0(X(x, \xi), \ \Xi(x, \xi), \ T(x, t, \xi))$, i.e. find a function T solution of a standard non-linear equation in t, depending smoothly on the parameters (x, ξ). We first solve it for $|\,\xi\,| = 1$, with $T(n) = t(n)$, and then extend T as a homogeneous function in ξ of degree 0, so that T exists in a fixed interval in t, near n.

The functions (x, Ξ, T) define a mapping $\chi\colon \Sigma \to \Sigma$; we extend it to a symplectic mapping $\tilde{\chi}$ and consider F, elliptic Fourier integral operator with canonical mapping $\tilde{\chi}$, as previously. Then $Q' = \Theta P^F$ has the same principal part as Q, and the same non zero $q_1'^0 = q_1{}^0$: by (iii), Theorem 1.1, $Q' \sim Q$, i.e. $Q \sim^F P$.

Assume now $p_1{}^0 \equiv 0$. We want to compute $I(\Theta P^F)$. First, $I(\Theta P^F)$ only depends on χ, and not on F: if G is another Fourier-integral operator with the same χ, then $Q^F = \Theta P^F = \Theta F^{-1} G \Theta^{-1} \Theta G^{-1} P G G^{-1} F = (\Theta F^{-1} G \Theta^{-1})\ Q^G (G^{-1} F)$, i.e. $Q^F \sim Q^G$, hence $I(Q^F) = I(Q^G)$. As above, we can write $P = H^2 + D$, with $d_0{}^0 = I(P)$. Then $Q^F = \Theta(F^{-1} H F)^2 + \Theta F^{-1} D F$, and $(\Theta F^{-1} D F)_0^0 = (1/\theta^2)\, I(P)$ $(\chi\,|_\Sigma)$. It is well-known that there is an elliptic operator E such that $E^-(F^{-1} H F) E = \theta D_t$, where θ stands for the operator with symbol θ; this means that we can assume, without restrictions, F already chosen with $F^{-1} H F = \theta D_t$. We write $\Theta(F^{-1} H F)^2 = \Theta \theta D_t \theta D_t = \Theta \theta^2 D_t{}^2 + \Theta \theta [D_t \theta] D_t = R$, and (usual notations) $r_0 = \tau^2$, $r_1 = -i\tau(\theta_t/\theta) + \tau^2(\cdots)$, $r_2 = \tau^2(\cdots) + \tau(\cdots)$; this implies $I(R) = r_2{}^0 - \frac{1}{4}(r_1{}^1)^2 + (i/2) r_{1t}^1 = \frac{1}{4}(\theta_t'/\theta)^2 + \frac{1}{2}(\theta_t'/\theta)_t'$, finally $I(Q^F) = \frac{1}{4}(\theta_t'/\theta)^2 + \frac{1}{2}(\theta_t'/\theta)_t' + (1/\theta^2)\, I(P) \circ (\chi\,|_\Sigma)$. Let us recall that here θ stands for $\theta\,|_\Sigma = (1/T_t')$.

We give here some simple applications of the above formula. If $I(P) \in \mathbb{R}$, we can choose θ such that $I(Q^F) = 0$: this is done simply by solving an ordinary differential equation for T, with data $T(n) = t(n)$, $T_t'(n) = 1$ (for instance), and T homogeneous of degree 0 in ξ (we take here $X = x$, $\Xi = \xi$).

If not, we can make $\operatorname{Re} I(Q^F) \equiv 0$ by the preceding argument. If $I(P)$ and $I(Q^F)$ are both purely imaginary, necessarily we have $\frac{1}{2}(\theta_t'/\theta)^2 + (\theta_t'/\theta)_t' = 0$, $I(Q^F) = (1/\theta^2)\, I(P)(\chi\,|_\Sigma)$. Setting $u = (\theta_t'/\theta)$, the first equation has solution $u = 0$, or $u = (2/t - t_0)$, hence $\theta = cte$ or $\theta = C(t - t_0)^2$ (C and t_0 are real constants); this gives curious conditions for the equivalence, and proves the assertions in Theorem 1.2.

3. Hyperbolic Symplectic case (codim 2)

For the hyperbolic case, we take $p_0 = t\tau$. Then $\Sigma = \{t = \tau = 0\}$, and

$$(1) \quad -\frac{1}{i}(t\partial - \tau\partial_\tau)_{''0} + t\tau(a_1 - b_1) + a_0(p_1 - q_1) = 0$$

$$(2) \quad -\frac{1}{i}(t\partial_t b_1 - \tau\partial_\tau a_1) + a_0(p_2 - q_2) + t\tau(a_2 - b_2) + a_1 p_1 - q_1 b_1$$

$$+ \frac{1}{i}(a_0 \xi p_{1x} + a_{0\tau} p_{1t} - q_{1\xi} a_{0x} - q_{1\tau} a_{0t}) = 0$$

As before, we have, if $P \sim Q$, $p_1{}^0 = q_1{}^0$. To solve an equation of the form $L(1) = f$, it is necessary to have $f|_\Sigma = 0$; conversely, if $f|_\Sigma = 0$, we can find $a_1 - b_1$ and a_0 such that, with these values, $L(1) = f$ (see Hanges [10] for details). This proves (i), Theorem 2.1.

Assume now that a_0, $a_1 - b_1$ satisfy $L(1) = 0$. As before, we set $\psi^{j,k} = (1/j!k!)(\partial^{s+k}/\partial t^j\, \partial\tau^k)\,\psi\,|_\Sigma$. Then $(L(2))^{0,0} = a_0{}^0(p_2 - q_2)^0 + p_1{}^0(a_1 - b_1)^0 - (1/i)\, H_{p_1{}^0} a_0{}^0 + (1/i)\, P_1^{1,0} a_0^{0,1} - (1/i)\, q_1^{0,1} a_0^{1,0}$. From (1), $-(1/i)(t\partial_t - \tau\partial_\tau)(a_0{}^0 + t a_0^{1,0} + \tau a_0^{0,1} + \cdots) + t\tau(a_1 - b_1) + (p_1 - q_1)a_0 = 0$, we get $i a_0^{1,0} = -a_0{}^0(p_1 - q_1)^{1,0}$, $i a_0^{0,1} = +a_0{}^0(p_1 - q_1)^{0,1}$, $(a_1 - b_1)^0 = -a_0{}^0(p_1 - q_1)^{1,1} - a_0^{1,0}\,(p_1 - q_1)^{0,1} - a_0^{0,1}(p_1 - q_1)^{1,0}$, hence $(L(2))^0 = -(1/i)\, H_{p_1{}^0} a_0{}^0 + a_0{}^0[(p_2 - q_2)^0 - p_1{}^0(p_1 - q_1)^{1,1} - p_1{}^0(p_1 - q_1)^{0,1}\, i(p_1 - q_1)^{1,0} + p_1{}^0(p_1 - q_1)^{1,0}\, i(p_1 - q_1)^{0,1} - p_1^{1,0}(p_1 - q_1)^{0,1} - q_1^{0,1}(p_1 - q_1)^{1,0}]$, $(L(2))^0 = -(1/i)\, H_{p_1{}^0} a_0{}^0 + a_0{}^0[I(P) - I(Q)]$, where $I(P) = p_2{}^0 - p_1{}^0 p_1^{1,1} - p_1^{1,0} p_1^{0,1}$. If $p_1{}^0 = cte$, then $P \sim Q \Rightarrow I(P) = I(Q)$, which proves (iii).

Assume now that $|\,\xi\,|\, H_{p_i{}^0}$ is H-solvable (see definition in part I.2)). Then we can choose $a_0{}^0 \neq 0$, homogeneous of order 0, such that $(L(2))^0 = 0$. Now we solve (1) with $a_0|_\Sigma = a_0{}^0$ just obtained, and compute $(L(3))^0$: $(L(3))^0 = -(1/i)\, H_{p_1{}^0} a_1{}^0 + a_1{}^0[I(P) - I(Q)] +$ known terms. We choose $a_1{}^0$ so that $(L(3))^0 = 0$, and then solve (2) with $a_1|_\Sigma = a_1{}^0$, and so on. This prove (ii), Theorem 2.1.

Let $P = P_0 + P_1 + P_k + \cdots$, $Q = P_0 + P_1 + Q_k + \cdots$ $(k \geqslant 2)$, with all the lower order terms in P and Q tangential (which is no restriction, according to (i)), and $p_1 = cte$, so that P_1 is simply the multiplication by a constant. Then $(A_0 + A_1 + \cdots)(P_0 + P_1 + P_k + \cdots) = (P_0 + P_1 + Q_k + \cdots)(A_0 + B_1 + \cdots)$ gives

$$\underbrace{A_0 P_0 - P_0 A_0} + \underbrace{A_0 P_1 - P_1 A_0 - P_1 A_0 + A_1 P_0 - P_0 B_1} + \cdots$$

$$+ \underbrace{P_1(A_l - B_l) + A_{l+1} P_0 - P_0 B_{l+1}} + \cdots$$

$$+ \underbrace{A_0 P_k - Q_k A_0 + A_{k-1} P_1 - P_1 B_{k-1} + A_k P_0 - P_0 B_k} + \cdots = 0,$$

hence

$$(l+1) \text{ is } -\frac{1}{i}(tb'_{l_t} - \tau a'_{l_\tau}) + p_1(a_l - b_l) + p_0(a_{l+1} - b_{l+1}) = 0 \qquad (0 \leqslant l \leqslant k-2),$$

$$(k) \text{ is } -\frac{1}{i}(tb'_{k-1_t} - \tau a'_{k-1_\tau}) + p_1(a_{k-1} - b_{k-1}) + p_0(a_k - b_k) + a_0(p_k - q_k) = 0,$$

$$(k+1) \text{ is } -\frac{1}{i}((tb'_{k_t} - \tau a'_{k_\tau}) + p_1(a_k - b_k) + p_0(a_{k+1} - b_{k+1}) + a_0(p_{k+1} - q_{k+1}) + p_k a_1 - q_k b_1 - \frac{1}{i} H_{p_k} a_0 + \frac{1}{i}(p_k - q_k)_\xi\, a_0 x = 0.$$

Here, (1) is just $-(1/i)(t\partial_t - \tau\partial_\tau)a_0 + t\tau(a_1 - b_1) = 0$, so that $(a_1 - b_1)^{m,m} = 0$ $(m \geqslant 0)$. Identifying the terms in $(l+1)$ (for $l + 1 < k$), we obtain $(a_{l+1} - b_{l+1})^{m-1,m-1} = -(p_1 + m/i)(a_l - b_l)^{m,m}$ $(m \geqslant 1)$. This implies $(a_l - b_l)^0 = 0$, $0 \leqslant l \leqslant k-1$. Thus, if $P \sim Q$, necessarily, $p_k = q_k$.

Assume now that $p_k = q_k$ and H_{p_k} is H-solvable, and let us examine the sufficiency of this:

For each equation $(l+1)$ $(0 \leqslant l \leqslant k-1)$, we can take $a_l - b_l = 0$, $a_{l+1} - b_{l+1} = 0$, $a_l = a_l^0$ arbitrary, to be chosen. Then $(L(k+1))^0 = iH_{p_k}a_0^0 + a_0^0(p_{k+1} - q_{k+1})^0$, and we can choose $a_0^0 \neq 0$ such that $(L(k+1))^0 = 0$; after this we solve $(k+1)$, and it is important to notice that a_k^0 is free, and $a_{k+1} - b_{k+1}$ depends only on P, Q and a_0^0. Similarly, $(L(k+2))^0 = p_1(a_{k+1} - b_{k+1})^0 + p_{k+1}^0 a_1 - q_{k+1}^0 b_1 + a_0(p_{k+2} - q_{k+2})^0 + iH_{p_k}a_1^0 +$ terms depending on a_0; thus, we can choose a_1^0 such that $(L(k+2))^0 = 0$, and we solve $(k+2)$ with a_{k+1}^0 free and $a_{k+2} - b_{k+2}$ depending only on P, Q, a_0, a_1.

The general equation (l) is $(l \geqslant k+2)$

(l) $i(tb'_{l-1t} - \tau a'_{l-1\tau} + p_0(a_l - b_l) + p_1(a_{l-1} - b_{l-1}) + p_k(a_{l-k} - b_{l-k}) + p_{k+1}a_{lk-1} - q_{k+1}b_{l-k-1} + \cdots + a_0(p_l - q_l) + iH_{p_k}a_{l-k-1} +$ (terms depending on P, Q and $a_j, b_j, j \leqslant l-k-2) = 0$. If we assume that each $a_j - b_j$ is known $(j \leqslant l-1)$ and depends only on P, Q, and the a_i, b_i for $0 \leqslant i \leqslant j-k-1$, then we can choose a_{l-k-1}^0 making $(L(l))^0 = 0$, and solve (l) with a_{l-1}^0 free and $a_l - b_l$ depending only on $P, Q, a_i, b_i, i \leqslant l-k-1$.

This shows that $a_0^0, \ldots, a_{k-1}^0$ can be successively used to make $(k+1), \ldots, (2k)$ solvable, then a_k can be used to make $(2k+1)$ solvable, and so on. This proves (iv), Theorem 2.1.

We study now the conjugation by Fourier-integral operators.

LEMMA. *Let $\mathscr{T}$, T, homogeneous of degree* 1 *and* 0 *be given, with* $\{\mathscr{T}, T\} = 1$, *and* $\mathscr{T} = T = 0$ *on* Σ; *assume that* $X(x, \xi)$ *and* $\Xi(x, \xi)$, *homogeneous of degree* 0 *and* 1, *define on* Σ *a symplectic mapping* χ. *Then we can extend* (X, Ξ) *to* $(\tilde{X}, \tilde{\Xi})$ *near* Σ *such that the mapping* $\tilde{\chi}$: $(x, t, \xi, \tau) \to (\tilde{X}, T, \tilde{\Xi}, \mathscr{T})$ *is symplectic and* $\tilde{X}\,|_\Sigma = \chi$.

For the mappings $\tilde{\chi}$ we are considering here, we must have, moreover $p_0 \circ \tilde{\chi} = ep_0$, i.e. $\mathscr{T}T = et\tau$; this is possible only if $\alpha' = \beta = 0$ on Σ, or $\beta' = \alpha = 0$ on Σ: in the first case, $e\,|_\Sigma = \alpha\beta'\,|_\Sigma = 1$, in the second $e\,|_\Sigma = \beta\alpha'\,|_\Sigma = -1$.

Let F be an elliptic Fourier integral operator with canonical transformation $\tilde{\chi}$, and $P^F = F^{-1}PF$. On Σ, the subprincipal symbols compose, i.e. $(P^F)^s = (P^s) \circ \chi$; moreover, $(P^F)^s = p_1^F - (1/2i)e$ on Σ. If E is elliptic of symbol $1/e$, then, for $Q = EP^F$, $q_0 = p_0 = t\tau$, $q_1^0 = P_1^F/e = (1/e)[p_1^0 \circ \chi - 1/2i + e/2i]$. In the first case, $q_1^0 = p_1^0 \circ \chi$, in the second $q_1^0 = -p_1^0 \circ \chi - i$.

4. Elliptic Symplectic case (codim 2)

Here, we take $p_0 = \tau^2 + t^2\xi_1^2$, $\Sigma = \{t = \tau = 0\}$, and we use the same notations as in the hyperbolic case.

The first equation is

$$2i(\tau\partial_t - t\xi_1^2\partial_\tau + t^2\xi_1\partial_1)a_0 + p_0(a_1 - b_1) + a_0(p_1 - q_1) = 0. \tag{1}$$

Set $H = \tau\partial_t - t\xi_1^2\partial_\tau$: we have necessarily $p_1^0 = q_1^0$, and $H(t^k\tau^l) = t^{k-1}\tau^{l-1}(k\tau^2 - l\xi_1^2t^2) = t^{k-1}\tau^{l-1}(k\tau^2 + kt^2\xi_1^2 - (k+l)t^2\xi_1^2) = kp_0t^{k-1}\tau^{l-1} - (k+l)\,\xi_1^2t^{k+1}\tau^{l-1} = -lp_0t^{k-1}\tau^{l-1} + (k+l)\,t^{k-1}\tau^{l+1}$, and $H(t^k) = k\tau t^{k-1}$, $H(\tau^l) = -\xi_1^2lt\tau^{l-1}$. This implies that we can solve (1) formally, by determining successively the terms in the Taylor expansion of a_0 with respect to (t, τ), with a_0^0 free; then we can absorb any flat (on Σ) right hand side ψ by adding to $a_1 - b_1$ the flat function ψ/p_0. In this process, $a_1 - b_1$ is chosen along with a_0, and both depend on a_0^0.

From the preceding point, to solve an equation of the form $L(1) = f$, it is necessary and sufficient to have $f\,|_\Sigma = f^0 = 0$. Thus, by a standard procedure, we can find $\tilde{P} \sim P$, with $\tilde{p}_i = \tilde{p}_i^0$ for all $i \geqslant 1$.

Assume that the lower order terms of P and Q are already "tangential," and let us compute $(L(2))^0$:

$$(L(2))^0 = a_0^0(p_2 - q_2)^0 + p_1(a_1 - b_1)^0 + iH_{p_1}a_0^0 - \xi_1^2a_0^{0,2} + a_0^{2,0}.$$

From (1), we have $-2i\xi_1^2a_0^2 = 0$, $2i\partial_ta_0^1 = 0$, $2ia_0^{1,1} + (a_1 - b_1)^0 = 0$, $-2i\xi_1^2a_0^{1,1} + 2i\xi_1\partial_1a_0^0 + \xi_1^2(a_1 - b_1)^0 = 0$, $4ia_0^{2,0} - 4i\xi_1^2a_0^{0,2} = 0$. Hence

$$\begin{aligned}(L(2))^0 &= a_0^0(p_2 - q_2) - i\frac{p_1}{\xi_1}\partial_1a_0^0 + iH_{p_1}a_0^0 \\ &= a_0^0(p_2 - q_2) + i\xi_1H_{(p_1/\xi_1)}a_0^0.\end{aligned}$$

Thus, if $p_1/\xi_1 = cte$, necessarily, $p_2 = q_2$; if Hp_1/ξ_1 is H-solvable, $p_1 = q_1$ is sufficient for $P \sim Q$. The computations are analogous to those in the hyperbolic case, with p_{1/ξ_1} replacing p_1.

Let us check briefly the action of Fourier-integral operators. The mappings χ: $(x, t, \xi, \tau) \to (X, \Xi, T, \mathscr{T})$ such that $\chi(\Sigma) \subset \Sigma$ and $p_0 \circ \chi = ep_0$ must be of the form $T = \alpha t + \beta\tau + \cdots$, $\mathscr{T} = \alpha' t + \beta'\tau + \cdots$, with $\left(\begin{smallmatrix}\alpha & \beta\\ \alpha' & \beta'\end{smallmatrix}\right)$ of the form

$$\begin{pmatrix} \epsilon\lambda\xi_1 \dfrac{\cos\theta}{\Xi_1} & -\epsilon\lambda \dfrac{\sin\theta}{\Xi_1} \\ \lambda\xi_1 \sin\theta & \lambda\cos\theta \end{pmatrix} \quad (\theta \in \mathbf{R},\ \epsilon = \pm 1,\ \lambda \text{ homogeneous of degree } 0).$$

Moreover, $\{\mathscr{T}, T\} = \beta'\alpha - \alpha'\beta = +\epsilon\lambda^2(\xi_1/\Xi_1) = +1$, and $e = \lambda^2$, so that, for $Q = EF^{-1}PF$ (usual notations), $q_0 = p_0$, $q_1{}^0 = \epsilon(\xi_1/\Xi_1)(p_1{}^0 \circ \chi)$, i.e. $q_1{}^0/\xi_1 = \epsilon(p_1{}^0/\xi_1) \circ \chi$. This is analogous to B), Theorem 2.1, except that we cannot "exchange the factors" as in the hyperbolic case. Finally, χ must map n onto n, so that $\epsilon = +1$.

5. Hyperbolic involutive case (codim 2)

We take $p_0 = \xi\eta$ (the coordinates are $(x, y, z, \xi, \eta, \zeta)$). The first equation is

$$i(\eta\partial_x + \xi\partial_y)a_0 + \xi\eta(a_1 - b_1) + a_0(p_1 - q_1) = 0. \tag{1}$$

As usual, if $P \sim Q$, we must have $p_1{}^0 = q_1{}^0$.

Set for a moment $a_1 - b_1 = i\mu$, $p_1 - q_i = i\lambda$; expanding the functions in (1) in Taylor Series in (ξ, η), we obtain, for the modified equation $(1/i)L(1) = f$, the relations $f^0 = 0$,

$$\Delta_1 a_0{}^0 \equiv \partial_y a_0{}^0 + \lambda^1 a_0{}^0 = f^1, \qquad \Delta_2 a_0{}^0 \equiv \partial_x a_0{}^0 + \lambda^2 a_0{}^0 = f^2,$$
$$\Delta_2 a_0{}^2 + a_0{}^0\lambda^{0,2} = f^{0,2}, \qquad \Delta_1 a_0{}^1 + a_0{}^0\lambda^{2,0} = f^{2,0},$$
$$\Delta_1 a_0{}^2 + \Delta_2 a_0{}^1 = -\mu^0 - a_0{}^0\lambda^{1,1} + f^{1,1}.$$

For the higher order terms, we have

$$\Delta_1 a_0^{m,0} = f^{m+1,0} + \text{known terms}, \qquad \Delta_2 a_0^{0,m} = f^{0,m+1} + \text{known terms},$$

and all the other relations involving terms homogeneous in (ξ, η) of degree $m + 1$ can be satisfied, provided that the terms of degree $m - 1$ in μ are chosen conveniently.

For a non-zero solution a_0 of (1) to exist, we must have $\Delta_1 a_0{}^0 = 0$, $\Delta_2 a_0{}^0 = 0$, which implies $\partial_x\lambda^1 = \partial_y\lambda^2$ (i.e. Δ_1 and Δ_2 commute).

From now on, we assume this condition; then the preceding point shows that to solve $L(1) = f$, it is necessary and sufficient to have $f^0 = 0$, $\Delta_2 f^1 = \Delta_1 f^2$.

To operate the standard reduction of P to $\tilde{P}$ (see (i) in Theorem 1.1, Theorem 2.1 and Theorem 2.2), we have to choose, in (l), $\tilde{P}_l$ in such a way that $a_0{}^0(p_l - \tilde{p}_l)^0 =$ known quantity,

$$\Delta_2[a_0{}^1(p_l - \tilde{p}_l)^0 + a_0{}^0(p_l - \tilde{p}_l)^1] - \Delta_1[a_0{}^2(p_l - \tilde{p}_l)^0 + a_0{}^0(p_l - \tilde{p}_l)^2] = \text{known qu.}$$

But we have, from (1), $\Delta_1 a_0{}^0 = \Delta_2 a_0{}^0 = 0$, so that the relations are $(p_l - \tilde{p}_l)^0 =$ kn. qu., $\partial_x(p_l - \tilde{p}_l)^1 - \partial_y(p_l - \tilde{p}_l)^2 =$ known qu. They can clearly be satisfied by taking $\tilde{p}_l = \tilde{p}_l{}^0 + \tilde{p}_l{}^1\xi$, with $\tilde{p}_l$ and $\tilde{p}_l{}^1$ conveniently chosen, which proves (i).

The second equation is here

$$(2)\quad -\frac{1}{i}(\eta\partial_x + \xi\partial_y)b_1 + b_1(p_1 - q_1) + \xi\eta(a_2 - b_2) = -a_0(p_2 - q_2) - p_1(a_1 - b_1) + \frac{1}{i}q_{1\xi}a_{0x} + \frac{1}{i}q_{1\eta}a_{0y} + \frac{1}{i}q_{1\zeta}a_{0z} - \frac{1}{i}p_{1x}a_{0\xi} - \frac{1}{i}p_{1y}a_{0\eta} - \frac{1}{i}p_{1z}a_{0\zeta} - a_{0xy}\,.$$

We denote the right hand side by f, so that the conditions for (2) to have a solution are $f^0 = 0$, $\Delta_2 f' = \Delta_1 f^2$. Let us check the condition $f^0 = 0$, for instance. We have

$$-f^0 = +a_0{}^0(p_2 - q_2)^0 + p_1{}^0(a_1 - b_1)^0 - \frac{1}{i}H_{p_1{}^0}a_0{}^0 - \frac{1}{i}q_1{}^2a_{0y}^0 - \frac{1}{i}q_1{}^1a_{0x}^0 + \frac{1}{i}p_{1x}^0a_0{}^1 + \frac{1}{i}p_{1y}^0a_0{}^2 + a_{0xy}^0\,,$$

where $H_{p_1{}^0}$ denotes the Hamiltonian field of $p_1{}^0$ as a function of (z, ζ) only. Let us denote by α a fixed non zero solution of $\Delta_1\alpha = \Delta_2\alpha = 0$. Then $a_0{}^0$ is of the form $a_0{}^0 = E(z, \zeta)\alpha$, E being arbitrary. From the other relations in (1), we obtain, with $a_0{}^2 = C\alpha$, $a_0{}^1 = D\alpha$, $\Delta_2 a_0{}^2 = \alpha C'_x = -E\alpha\lambda^{0,2}$, $\Delta_1 a_0{}^1 = \alpha D'_y = -E\alpha\lambda^{2,0}$, and $(a_1 - b_1)^0 = i\mu^0 = -i\Delta_1 a_0{}^2 - i\Delta_2 a_0{}^1 - ia_0{}^0\lambda^{1,1} = -i\alpha C'_y - i\alpha D'_x - iE\alpha\lambda^{1,1}$. Thus, $-f^0 = -i\alpha[(p_1{}^0C)'_y + (p_1{}^0D)'_x] + G$, where $G = E\alpha(p_2 - q_2)^0 - ip_1{}^0\alpha E\lambda^{1,1} + iH_{p_1{}^0}a_0{}^0 + iq_1{}^1a_{0x}^0 + iq_1{}^2a_{0y}^0 + a_{0xy}^0$, and $C'_x = -E\lambda^{0,2}$, $D'_y = -E\lambda^{2,0}$. The following lemma is concerned with systems of the form just obtained.

LEMMA. *Let* (*) *be the system* (*in two variables* (x, y))

$$(*)\quad \begin{cases} (pC)'_y + (pD)'_x = F \\ C'_x = R,\ D'_y = S \end{cases}$$

where p is a smooth given non-zero function, F, R and S are smooth given functions, and C and D are the unknown functions.

Set $Lp = (\text{Log } p)''_{xy}$, *and* $p \vee F = (F/p)''_{xy} - R''_{yy} - (R(p'_y/p))'_y - S''_{xx} - (S(p'_x/p))'_x$. *Then* $(*)$ *implies the systems*

$$*_k \begin{cases} (p_k C)'_y + (p_k D)'_x = F_k \\ C'_x = R,\ D'_y = S \end{cases} \qquad k \geqslant 1,$$

where $p_k = L^k p$, $F_k = L^{k-1}p(\vee \cdots \vee (Lp \vee (p \vee F)) \cdots)$. *In general, there are finitely many necessary and sufficient conditions on F, R and S for $(*)$ to be solved. If $L^k p \equiv 0$ for some $k \geqslant 1$, $F_k \equiv 0$ is a necessary condition. For $k = 1$, it is also sufficient ($L_p = 0$ means that p is of the form $a(x)\, b(y)$).*

Proof. We rewrite the first equation in the form

$$C_y + C\frac{p'_y}{p} + D_x + D\frac{p'_x}{p} = \frac{F}{p}.$$

We take now the xy-derivative and obtain

$$R''_{yy} + \left(R\frac{p'_y}{p}\right)'_y + (zC)'_y + S''_{xx} + \left(S\frac{p'_x}{x}\right)'_x + (zD)'_x = \left(\frac{F}{p}\right)''_{xy},$$

with

$$z = \left(\frac{p'_y}{p}\right)'_x = \left(\frac{p'_x}{p}\right)'_y = (\text{Log } p)''_{xy} = Lp,$$

we obtain a new system $(*)_1$ analogous to $(*)$, with z instead of p, and

$$F_1 = \left(\frac{F}{p}\right)''_{xy} - R''_{yy} - \left(R\frac{p'_y}{p}\right)'_y - S''_{xx} - \left(S\frac{p'_x}{p}\right)'_x$$

instead of F.

If $z \equiv 0$ (i.e. p is of the form $a(x)\, b(y)$), we must have $F_1 \equiv 0$, and this is sufficient for $(*)$ to be solved; in fact, set $f_1 = C'_y + (b'/b)C$, $f_2 = D'_x + (a'/a)D$: we want to solve the two systems $\{C'_y + (b'/b)C = f_1,\ C'_x = R,\ D'_x + (a'/a)D = f_2,\ D'_y = S$, for f_1 and f_2 satisfying $F/p = f_1 + f_2$. The condition $F_1 \equiv 0$ implies that F/p can be written $f_1 + f_2$, with $f'_{1x} = R'_y + (b'/b)R$, $f'_{xy} = S'_x + (a'/a)S$; but these last equations are precisely the compatibilities required to solve the two above systems.

If $z \neq 0$, we can rewrite $(*)_1$ and obtain as above a new system $(*)_2$, with Lz instead of z, and a new F_2. We can, in general, determine C and D in terms of F, R and S from the first equations of $(*)$, $(*)_1$ and $(*)_2$. This gives a finite number of necessary and sufficient conditions on F, R and S for $(*)$ to be solved.

In our present case, G is of the form $G = E\beta + iH_{p_1{}^0}a_0{}^0$, where β denotes a function depending only on α, P, Q. Thus $p_1{}^0 \vee G/i\alpha = H_{(Lp_1{}^0)}E + E\beta_1$ (β_1 depends on α, P, Q), and more generally, $F_k = H_{L^k p_1{}^0}E + E\beta_k$. If $L^k p_1{}^0 \equiv 0$ for some k, we have, according to the Lemma, the necessary condition $\beta_k \equiv 0$; for $k = 1$, we find the condition $I_1(P) = I_1(Q)$, with

$$\begin{aligned} I(P) = (p_1^{1,1})xy &- \left(\frac{p_2{}^0}{p_1{}^0}\right) xy + \frac{1}{b} H_b(p_1{}^2)_y + \frac{1}{a} H_a(p_1{}^2)_y + H_{a'/a}p_1{}^1 \\ &+ H_{b^1/b}p_1{}^2 + \left(\frac{p_1{}^1 p_1{}^2}{p_1{}^0}\right)_{xy} - p_{1yy}^{0,2} - p_{1xx}^{2,0} \\ &- \left(\frac{b_1}{b}\, p_1^{0,2}\right)_y - \left(\frac{a_1}{a}\, p_1^{2,0}\right)_x \end{aligned}$$

(here, $p_1{}^0$ is of the form $p_1{}^0(x, y, z, \zeta) = a(x, z, \zeta)\, b(y, z, \zeta)$).

In general, it is not possible to satisfy the compatibility conditions mentioned in the lemma by chosing E conveniently, because E is a function of (z, ζ) only; this gives other conditions on P and Q.

The condition $\Delta_2 f' = \Delta_1 f^2$ on the right hand side of (2) can be analyzed by analogous (lengthy) computations.

Assume that we have the Levi Condition $p_1{}^0 \equiv 0$, then $-f^0 = a_0{}^0(p_2 - q_2)^0 + iq_1{}^1 a_{0x}^0 + iq_1{}^2 a_{0y}^0 + a_{0xy}^0$, and from (1), we get $-f^0 = a_0{}^0[(p_2 - q_2)^0 + q_1{}^1 q_1{}^2 - p_1{}^1 p_1{}^2 - \lambda_x{}^1]$. Thus $P \sim Q$ implies $I_2(P) = I_2(Q)$, with $I_2(P) = P_2{}^0 - p_1{}^1 p_1{}^2 + ip_{1x}^1$. No real simplification occurs in the computation of the other conditions.

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